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Covariant Wave Functions with Invariant Helicity for Massless Particles

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Abstract

Considering a wave function ψ for a massless particle, transforming according to an arbitrary irreducible representation (IR) of the homogeneous Lorentz group, we determine the basic conditions for ψ to be an eigenfunction with a specified value λ of the helicity in *all* Lorentz frames. The method used is direct and elementary, requiring no knowledge of the IR's of the Poincaré group. It is shown that there exists *no* invariant helicity state in unitary representations of the Lorentz group, and one such state in any non-unitary representation (with one extra in special cases).

1. *Introduction*

It is a well-known fact that in the group theoretical classification of elementary particles according to Wigner (1939) and Bargmann & Wigner (1948), massless particles are associated with irreducible representations (IR's) of the Poincaré group (PG) characterised by a vanishing value of $P^2 \equiv P^\mu P_\mu$ and a definite (integral or half integral) value of the helicity λ. The helicity for $P^2 = 0$, may be defined through the relation (Bargmann & Wigner, 1948) $W^\mu = \lambda P^\mu$ where $\{W^\mu\} = (\mathbf{J}.\mathbf{P}, P^0\mathbf{J} - \mathbf{P} \times \mathbf{K})$. Here $\mathbf{J}$ and $\mathbf{K}$ are the generators of rotations and boosts and the P^μ are the translation generators. The presence of only a single helicity in the massless case is in contrast with the case of massive particles where it is the spin s which (along with the squared mass) labels an IR, and all helicity values $s, s-1, \ldots, -s$ are necessarily present. Nevertheless, in describing massless particles it is not customary to use single component wave functions transforming according to the Wigner IR's (with their complicated momentum dependence); instead, one uses multicomponent wave functions transforming

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according to irreducible representations† of the homogeneous Lorentz group (HLG) generated by **J** and **K**. In doing so, one automatically introduces several values of the helicity. One question which arises immediately is whether these helicity values are all invariant, i.e. whether the description in terms of locally covariant wave functions is equivalent to the use of a direct sum of Wigner IR's corresponding to all the helicities involved. The earliest work of relevance to this question seems to be that of Hammer & Good (1957). In connection with a proposed wave equation for massless particles they showed by explicit evaluation of the helicity s part of a wave function transforming according to $D(0,s)$, that it remains a helicity s eigenstate in all inertial frames. Later it was emphasised by Shaw (1965) that the representation of the PG in the space of $D(0,s)$-type wave functions is reducible but indecomposable: the effect of a Lorentz transformation on a state of given helicity λ is to convert it into a superposition of states with helicity $\geqslant \lambda$. The only helicity state left invariant is then that with $\lambda = s$, in agreement with the result of Hammer & Good (1957).

One of our objectives in this paper is to generalise this result to any IR of the HLG. The basic conditions for a locally covariant wave function to have a definite value of helicity in all Lorentz frames will be derived below by a direct and elementary method. It follows trivially therefrom that in any finite dimensional IR, $D(m,n)$, the only invariant helicity is $\lambda = n - m$. Our work complements that of Weinberg (1964) and Frishman & Itzykson (1969) who have proved the converse result, namely that from entities transforming according to the Wigner IR belonging to a specified helicity λ, one can construct only such locally covariant fields as have the transformation property $D(m, m+\lambda)$, m being arbitrary. Unlike in their work, we do not need to invoke any knowledge of the Wigner IR's. In the case of *unitary* IR's of the HLG, we show that the formal result of Frishman & Itzykson (1969) namely that there are invariant helicity states with $\lambda = \pm j_0$, is essentially an empty statement because, as we prove below, no such state is normalisable. In any non-unitary IR, there is one invariant helicity state in general, and two in certain special cases.

2. *The Invariant Helicity State*

Consider a wave function which, at some physical space-time point, is given by $\psi(x)$ and $\psi'(x') = \Lambda(L)\psi(x)$ in different frames related by the Lorentz transformation L. The $\Lambda(L)$ constitute a representation of the HLG, which is taken to be irreducible. We now pose our problem as

† The IR's of the HLG are identified by the notation (j_0, c) where the labels are defined in terms of the eigenvalues $j_0^2 + c^2 - 1$ and $ij_0 c$ of the Casimir operators $\mathbf{J}^2 - \mathbf{K}^2$ and $\mathbf{J}.\mathbf{K}$ respectively. (j_0 = non-negative integer or half-integer, c = any complex number.) In the case of finite dimensional (non-unitary) IR's, an alternative and more convenient notation is $D(m,n)$, with m and n defined through $\mathbf{M}^2 \to m(m+1)$ and $\mathbf{N}^2 \to n(n+1)$, where $\mathbf{M} = \frac{1}{2}(\mathbf{J} + i\mathbf{K})$ and $\mathbf{N} = \frac{1}{2}(\mathbf{J} - i\mathbf{K})$. Unitary IR's are infinite dimensional and belong to either the principal series (c = pure imaginary) or the supplementary series ($j_0 = 0, 0 \leqslant c \leqslant 1$).

follows: If $\psi(x)$ represents a state which has definite momentum $\mathbf{p}$ and helicity λ in one reference frame, so that

$$(\mathbf{J}.\mathbf{p})\,\psi(x)=\lambda p\psi(x) \tag{2.1}$$

under what conditions will the equation

$$(\mathbf{J}.\mathbf{p}')\,\psi'(x')=\lambda p'\,\psi'(x') \tag{2.2}$$

be valid in every other reference frame, $\mathbf{p}'$ being the momentum of the state in the new frame? Since rotations do not affect the helicity, what we have to worry about is the compatibility of (2.1) and (2.2) when an arbitrary boost is involved, i.e.

$$\psi'(x')=e^{i\alpha\hat{\mathbf{n}}.\mathbf{K}}\psi(x) \tag{2.3}$$

$\beta=\tanh\alpha$ being the velocity and $\hat{\mathbf{n}}$ the direction of the boost. Equation (2.2) may then be rewritten as

$$[e^{-i\alpha\hat{\mathbf{n}}.\mathbf{K}}(\mathbf{J}.\mathbf{p}')\,e^{i\alpha\hat{\mathbf{n}}.\mathbf{K}}]\,\psi(x)=\lambda p'\,\psi(x) \tag{2.4}$$

The operator in square brackets can be evaluated with the aid of the formula

$$e^{X}\,Ye^{-X}=Y+[X,Y]+\frac{1}{2!}[X,[X,Y]]+\ldots \tag{2.5}$$

The commutators involved in the present case are all expressible in terms of two basic ones:

$$[-i\mathbf{K}.\hat{\mathbf{n}},\mathbf{J}]=i(\hat{\mathbf{n}}\times i\mathbf{K}),\qquad [-i\mathbf{K}.\hat{\mathbf{n}},i\mathbf{K}]=i(\hat{\mathbf{n}}\times\mathbf{J}) \tag{2.6}$$

which in turn may be deduced from the algebra of the Lorentz generators. One finds then that

$$e^{-i\alpha\hat{\mathbf{n}}.\mathbf{K}}\,\mathbf{J}\,e^{i\alpha\hat{\mathbf{n}}.\mathbf{K}}=\mathbf{J}\cosh\alpha+i(\hat{\mathbf{n}}\times i\mathbf{K})\sinh\alpha-(\mathbf{J}.\hat{\mathbf{n}})\,\hat{\mathbf{n}}(\cosh\alpha-1) \tag{2.7}$$

Combining this with the explicit expression for the transformed momentum $\mathbf{p}'$ in terms of $\mathbf{p}$ namely

$$\mathbf{p}'=\mathbf{p}+(\hat{\mathbf{n}}.\mathbf{p})\,\hat{\mathbf{n}}(\cosh\alpha-1)-\hat{\mathbf{n}}p\sinh\alpha \tag{2.8}$$

one finally obtains the condition (2.4) as

$$[(\mathbf{J}.\mathbf{p})\cosh\alpha+i(\hat{\mathbf{n}}\times i\mathbf{K}).\mathbf{p}\sinh\alpha-(\mathbf{J}.\hat{\mathbf{n}})p\sinh\alpha]\,\psi(x)$$
$$=\lambda[p\cosh\alpha-(\hat{\mathbf{n}}.p)\sinh\alpha]\,\psi(x) \tag{2.9}$$

To see the import of this equation it is convenient to choose the direction of $\mathbf{p}$ to be the z-axis, when (2.9) simplifies to

$$[J_3(\cosh\alpha-n_3\sinh\alpha)-n_1(J_1+K_2)\sinh\alpha-n_2(J_2-K_1)\sinh\alpha]\,\psi(x)$$
$$=\lambda[\cosh\alpha-n_3\sinh\alpha]\,\psi(x) \tag{2.10}$$

We recall that this is nothing but equation (2.2) re-expressed in terms of quantities in the original reference frame. Its compatibility with equation (2.1), which now takes the form

$$J_3 \psi = \lambda \psi \tag{2.11a}$$

requires evidently that

$$(J_1 + K_2)\psi = 0 \qquad \text{and} \qquad (J_2 - K_1)\psi = 0$$

or equivalently,

$$M_- \psi = 0 \qquad \text{and} \qquad N_+ \psi = 0 \tag{2.11b}$$

where M_- and N_+ are ladder operators for $\mathbf{M} \equiv \frac{1}{2}(\mathbf{J} + i\mathbf{K})$ and $\mathbf{N} \equiv \frac{1}{2}(\mathbf{J} - i\mathbf{K})$.

The conditions (2.11) for a particular helicity eigenstate to be Lorentz invariant are precisely the ones obtained by far more elaborate methods starting from the Wigner IR's (Weinberg, 1964; Frishman & Itzykson, 1969). Determining the state singled out by (2.11) is now an almost trivial matter. In the case of a finite dimensional IR, $D(m,n)$, the commuting angular momentum like operators **M** and **N** operate in *independent* spaces, and (2.11b) then requires

$$M_3 \psi = -m\psi \qquad \text{and} \qquad N_3 \psi = n\psi \tag{2.12}$$

Since $J_3 = M_3 + N_3$ it follows that

$$\lambda = n - m \tag{2.13}$$

This argument cannot be applied to unitary representations, which are necessarily infinite dimensional and do not have **M** and **N** as independent operators, for the hermiticity of **J** and **K** implies that $\mathbf{M} = \mathbf{N}^\dagger$. However this relation, and in particular its consequence:

$$(M_\pm)^\dagger = N_\mp, \tag{2.14}$$

can now be exploited. We write each of the Casimir operators of the HLG, $\mathbf{J}^2 - \mathbf{K}^2$ and $\mathbf{J}.\mathbf{K}$, in two alternative forms:

$$\mathbf{J}^2 - \mathbf{K}^2 = (J_3{}^2 - K_3{}^2) + 2(M_+ M_- + N_- N_+ - iK_3) \tag{2.15a}$$

$$= (J_3{}^2 - K_3{}^2) + 2(M_- M_+ + N_+ N_- + iK_3) \tag{2.15b}$$

and

$$\mathbf{J}.\mathbf{K} = -i(M_+ M_- - N_- N_+ - J_3) + J_3 K_3 \tag{2.16a}$$

$$= -i(M_- M_+ - N_+ N_- + J_3) + J_3 K_3 \tag{2.16b}$$

It follows then that for a state ψ transforming locally like (j_0, c) and satisfying equations (2.11),

$$(j_0{}^2 + c^2 - 1)(\psi, \psi) = (\psi, (\mathbf{J}^2 - K^2)\psi)$$

$$= (\psi, [\lambda^2 - K_3{}^2 - 2iK_3]\psi) \tag{2.17a}$$

$$= (\psi, [\lambda^2 - K_3{}^2 + 2iK_3]\psi) \tag{2.17b}$$

and

$$ij_0\, c(\psi, \psi) = (\psi, [i\lambda + \lambda K_3]\,\psi) \tag{2.18a}$$

$$= (\psi, [-i\lambda + \lambda K_3]\,\psi) \tag{2.18b}$$

The hermiticity relation (2.14) has been used in arriving at (2.17b) and (2.18b) with the aid of (2.15b) and (2.16b). From equations (2.17) we immediately deduce that

$$(\psi, K_3\,\psi) = 0 \tag{2.19a}$$

and from equations (2.18),

$$(\psi, \lambda\psi) = 0 \qquad \text{or} \qquad \lambda = 0 \tag{2.19b}$$

Thus in a *unitary* representation, *no helicity eigenstate with* $\lambda \neq 0$ *can be invariant*. The necessary conditions for invariance can be sharpened by feeding equations (2.19) back into (2.17) and (2.18). We obtain

$$(j_0{}^2 + c^2 - 1)\,(\psi, \psi) = -(\psi, K_3{}^2\,\psi) \leqslant 0 \tag{2.20a}$$

and

$$ij_0\, c(\psi, \psi) = 0 \tag{2.20b}$$

It is evident that the only unitary IR's in which these conditions can be met are the IR's $(0, c)$ of the principal series‡ (c pure-imaginary) or of the supplementary series (c real, $0 \leqslant c \leqslant 1$).

There still remains the question whether in a given IR $(0, c)$ there exists a (normalizable) state ψ satisfying the constraints (2.11). To find an answer to this question it is necessary to expand ψ in terms of basis states $f_{j\sigma}$ which are simultaneous eigenstates of $\mathbf{J}^2$ and J_3 and whose behaviour under Lorentz transformations is known (Naimark, 1964; Frishman & Itzykson, 1969). The latter may be expressed, in a form which is most convenient for our purposes, as follows:

$$\begin{aligned} 2M_\pm f_{j\sigma} = &\pm ia_j[(j \mp \sigma)(j \mp \sigma + 1)]^{1/2} f_{j-1,\sigma\pm 1} \\ &+ (1 + ib_j)\,[(j \mp \sigma)(j \pm \sigma + 1)]^{1/2} f_{j,\sigma\pm 1} \\ &\mp ia_{j+1}[(j \pm \sigma + 1)(j \pm \sigma + 2)]^{1/2} f_{j+1,\sigma\pm 1} \end{aligned} \tag{2.21a}$$

$$\begin{aligned} 2M_3 f_{j\sigma} = &\; ia_j[j^2 - \sigma^2]^{1/2} f_{j-1,\sigma} + (1 + ib_j)\,\sigma f_{j\sigma} \\ &+ ia_{j+1}[(j+1)^2 - \sigma^2]^{1/2} f_{j+1,\sigma} \end{aligned} \tag{2.21b}$$

with

$$a_j = \left[\frac{(j^2 - j_0{}^2)(j^2 - c^2)}{j^2(4j^2 - 1)}\right]^{1/2}, \qquad b_j = \frac{ij_0\, c}{j(j+1)} \tag{2.22}$$

The effect of $N_\pm$, N_3 on $f_{j\sigma}$ are given by equations (2.21) with a_j, a_{j+1}, b_j replaced by their negatives.

‡ The IR (1,0) would also be allowed if one could have the equality sign in (2.20a), i.e. $K_3\psi = 0$, but it can be shown that this equation, taken together with (2.11), would force ψ to vanish identically.

In employing these properties in our problem, we note that ψ can be written as

$$\psi = \sum_j h(j) f_{j0} \tag{2.23}$$

since the helicity has to be necessarily zero according to (2.19). Further, the coefficient b_j vanishes in our case. It may be easily verified that on account of this, the recurrence relations for the $h(j)$ arising from the substitution of (2.23) in the two equations (2.11b), namely $M_{-}\psi = 0$ and $N_{+}\psi = 0$, coincide. We obtain thus a single three-term recurrence formula

$$a_j h(j-1) - i h(j) - a_{j+1} h(j+1) = 0 \tag{2.24}$$

Though one cannot obtain a simple solution for $h(j)$ from (2.24) it is not difficult to deduce its asymptotic behaviour as $j \to \infty$, which is all that matters as far as the normalisability of ψ is concerned. First, by dividing throughout by $h(j)$ one converts (2.24) into a recurrence relation for the ratio $h(j)/h(j-1)$. Substitution of the assumed asymptotic form $A + B/j + C/j^2 + \cdots$ for this ratio together with similar asymptotic forms obtained from (2.22) for the a_j, leads then to the result that for large j

$$i \frac{h(j)}{h(j-1)} \sim 1 + \frac{A}{j} + \cdots, \qquad A = \tfrac{1}{2} \pm c \tag{2.25}$$

Now, it is known (Bromwich, 1959) that as $j \to \infty$ the sequence $|h(j)|$ diverges if $\mathrm{Re}.A > 0$ and has a vanishing limit if $\mathrm{Re}.A < 0$. Further if $\sum_j |h(j)|^2$ is to converge, one must have $\mathrm{Re}.A < -\frac{1}{2}$. It follows therefore that the sequence $h(j)$ does not converge in any IR belonging to the principal series ($\mathrm{Re}.c = 0$). As far as IR's in the supplementary series are concerned, for $\frac{1}{2} < c < 1$ one can find solutions ψ with $h(j) \to 0$ (as $j \to \infty$) by taking $A = \frac{1}{2} - c$. However, even for these, the norm $(\psi, \psi) = \sum |h(j)|^2$ is infinite. Therefore no physically meaningful invariant helicity states exist when ψ transforms according to any unitary IR of the HLG.

Finally we observe that if ψ transforming according to any general IR of the HLG is considered, without insisting on unitarity or finite dimensionality, then one has the following as necessary conditions for invariant helicity:

$$\lambda (K_3 + i) \psi = i j_0 c \psi \tag{2.26a}$$

$$(K_3 + i)^2 \psi = (\lambda^2 - j_0^2 - c^2) \psi \tag{2.26b}$$

These are obtained by operating on ψ with $\mathbf{J}.\mathbf{K}$ and $\mathbf{J}^2 - \mathbf{K}^2$ in the forms (2.16a) and (2.15a), and using equations (2.11). Equations (2.26) in turn imply, for any $\lambda \neq 0$, that

$$(\lambda^2 - j_0^2)(\lambda^2 - c^2) = 0 \tag{2.27}$$

Actual determination of the state ψ is again done with the aid of a resolution of ψ similar to (2.23):

$$\psi = \sum_j h(j) f_{j\lambda}, \qquad \lambda = \epsilon j_0 \quad \text{or} \quad \epsilon c \tag{2.28}$$

where $\epsilon = \pm 1$. Application of equations (2.11), leads to two recurrence relations which can be solved to obtain (for $\lambda \neq 0$),

$$\frac{ih(j)}{h(j-1)} = \left[\frac{(2j+1)(j^2-\lambda^2)(j-j_0 c/\lambda)^2}{(2j-1)(j^2-j_0{}^2)(j^2-c^2)}\right]^{1/2} \tag{2.29a}$$

$$\to 1 + \left(\tfrac{1}{2} - \frac{j_0 c}{\lambda}\right)\frac{1}{j} + \cdots, \qquad \text{for } j \gg 1 \tag{2.29b}$$

Therefore $h(j)$ diverges as $j \to \infty$ unless Re. $[\frac{1}{2} - (j_0 c/\lambda)]$ is negative. Since the series in (2.28) could hardly be meaningful if the sequence $h(j)$ does not at least have a finite limit, we conclude that a state of invariant helicity $\lambda \neq 0$ cannot occur in IR's other than those with either

$$j_0 = \epsilon\lambda (= \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots) \qquad \text{and} \qquad \text{Re.}(\epsilon c) > \tfrac{1}{2} \tag{2.30a}$$

or

$$c = \lambda, \qquad \lambda - j_0 = \text{integer}, \qquad j_0 = 1, \tfrac{3}{2}, 2, \ldots \tag{2.30b}$$

It should be noted that equations (2.30) fix not only the magnitude but also the sign of the invariant helicity in any IR. In particular, the non-unitary IR's (1,1) and (1,−1) considered in the literature (Bender, 1968; Frishman & Itzykson, 1969) in connection with the radiation gauge treatment of the electromagnetic potentials, can accommodate only the helicities +1 and −1 respectively.‡

One last remark regarding the case $\lambda = 0$ which was excluded in the above discussion: by (2.26a) one has to have $j_0 c$ also vanishing and it can be verified then that the constraints on ψ reduce to the recurrence relation (2.24) on the coefficients $h(j)$. An analysis similar to that following equation (2.24) can then be carried through, and the final results turn out to be special cases of equations (2.30).

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‡ Both these representations would be inadmissible if we were to demand that $(\psi, \psi) = \sum |h(j)|^2$ be finite, for they do not satisfy the condition Re. $[\frac{1}{2} - (j_0 c/\lambda)] < -\frac{1}{2}$ which is necessary for convergence of this sum. However, since (ψ, ψ) is not Lorentz invariant in a *non-unitary* representation, it does not seem meaningful to require this property in general.

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